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ON SOME DISTRIBUTION PROBLEMS CONCERNING  
THE CHARACTERISTIC ROOTS OF (S SUB 1) (S SUB 2,  
SUB-1) UNDER VIOLATIONS

K. C. S. Pillai, et al

Purdue University

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### **ON SOME DISTRIBUTION PROBLEMS CONCERNING THE CHARACTERISTIC ROOTS OF $S_1 S_2^{-1}$ UNDER VIOLATIONS**

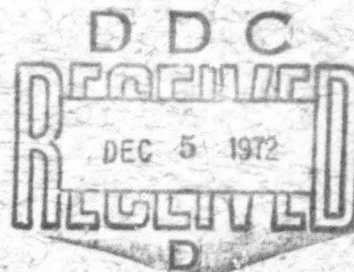
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<p>Pillai's distribution of the characteristic roots of <math>S_1 S_2^{-1}</math> under violations was used to obtain the following: the density function (under a condition), the moments and m.g.f. of <math>T</math> (a constant times Hotelling's <math>T_0^2</math>); the m.g.f. of Pillai's trace; the distribution of Wilks' <math>\Lambda</math>; and two expressions for the density function of the largest root. Earlier results of various authors on distribution theory concerning MANOVA and equality of two covariance matrices are shown to be special cases of the results of this paper.</p>		

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CONCERNING THE CHARACTERISTIC ROOTS  
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## FOREWORD

This is an interim report of the work done under Contract F33615-72-C-1400 of the Aerospace Research Laboratories, Air Force Command, United States Air Force. The work reported herein of K.C.S. Pillai was wholly and of Sudjana partly accomplished on Project 7071, "Research in Applied Mathematics", and was technically monitored by P. R. Krishnaiah of the Aerospace Research Laboratories. The work of Sudjana was in part supported by a Grant from the Ford Foundation, Program No. 36390.

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ON SOME DISTRIBUTION PROBLEMS CONCERNING THE  
CHARACTERISTIC ROOTS OF  $S_1 S_2^{-1}$  UNDER VIOLATIONS

by

K.C.S. Pillai and Sudjana

1. Introduction. The distribution of the characteristic roots of  $S_1 S_2^{-1}$  under certain violations was obtained by Pillai [12]. In the present paper Pillai's distribution has been used to derive the following: 1) the density function of  $T$ , a constant times Hotelling's  $T_0^2$ , 2) the moments of  $T$ , 3) the moment generating function of  $T$ , 4) the m.g.f. of Pillai's trace, 5) the distribution of Wilks' criterion and 6) the density function of the largest root in two forms. These results are useful in studying the exact robustness of at least two multivariate hypotheses, namely, a) MANOVA under violation of the assumption of a common covariance matrix and b) equality of covariance matrices of two  $p$ -variate normal populations when the normality assumption is disturbed. The exact robustness comparisons will be reported later.

2. The density function of  $T$ . In this section we will derive the density function of  $T = \lambda \operatorname{tr} S_1 S_2^{-1}$  where  $S_1$  ( $p \times p$ ) is distributed  $W(p, n_1, \Sigma_1, \Omega)$  i.e. non-central Wishart distribution on  $n_1$  d.f. with non-centrality  $\Omega$  and covariance matrix  $\Sigma_1$  and  $S_2$  ( $p \times p$ ) independently distributed central Wishart  $W(p, n_2, \Sigma_2, 0)$  where  $n_1, n_2 \geq p$  and  $\lambda$  is a positive real number.

In the case of  $n_1 < p$  we have only  $n_1$  non-zero roots of  $S_1 S_2^{-1}$  and the density function of  $T$  can be obtained from that for  $n_1 \geq p$  if in the latter case the substitutions  $(n_1, n_2, p) \rightarrow (p, n_1 + n_2 - p, n_2)$  are made.

Let us write  $A_1 = \frac{1}{2} \Sigma_1^{-1} S_1 \Sigma_1^{-1/2}$  and  $A_2 = \frac{1}{2} \Sigma_1^{-1/2} S_2 \Sigma_1^{-1/2}$ , then applying (29) of James [6] we get the joint density of  $A_1$  and  $A_2$  as:

$$C_1(p, n_2) e^{-\text{tr } \Lambda} |\Lambda|^{\frac{1}{2}n_2} e^{-\text{tr } A_2} |A_2|^{\frac{1}{2}(n_2-p-1)} |A_1|^{\frac{1}{2}(n_1-p-1)} \\ \cdot \int_{\text{Re}(Z)=X_0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} \exp(-\text{tr}(I-W)A_1) dZ$$

$$\text{where } C_1(p, n_2) = 2^{\frac{1}{2}p(p-1)} / [(2\pi i)^{\frac{1}{2}p(p+1)} \Gamma_p(\frac{1}{2}n_2)] ,$$

$\Lambda = \Sigma_1^{-1/2} \Sigma_2^{-1} \Sigma_1^{-1/2}$ ,  $W = \Sigma_1^{-1/2} Z^{-1} \Sigma_1^{-1/2}$  and  $Z = X_0 + iY$  with  $X_0$  p.d. symmetric matrix and  $Y$  a non-singular real symmetric matrix such that  $(I-W)$  is non-singular. Note that the roots are invariant under the above transformations

and also under the following transformation  $B_1 = (I-W)^{\frac{1}{2}} A_1 (I-W)^{\frac{1}{2}}$  and  $B_2 = (I-W)^{\frac{1}{2}} A_2 (I-W)^{\frac{1}{2}}$ . Using these, we obtain the joint density of  $B_1$  and  $B_2$  as follows:

$$(2.1) \quad C_1(p, n_2) e^{-\text{tr } \Lambda} |\Lambda|^{\frac{1}{2}n_2} e^{-\text{tr } B_1} |B_1|^{\frac{1}{2}(n_1-p-1)} |B_2|^{\frac{1}{2}(n_2-p-1)} \\ \cdot \int_{\text{Re}(Z)=X_0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} |I-W|^{-\frac{1}{2}(n_1+n_2)} \exp(-\text{tr}(I-W)^{-\frac{1}{2}} \Lambda (I-W)^{-\frac{1}{2}} B_2) dZ .$$

Now the Laplace transform of  $T = \lambda \text{tr } S_1 S_2^{-1} = \lambda \text{tr } B_1 B_2^{-1}$  is given by  $E(\exp(-t \lambda \text{tr } B_1 B_2^{-1}))$ . So, to find this transform multiply (2.1) by

$\exp(-t \lambda \operatorname{tr} B_1 B_2^{-1})$  and integrate  $B_1$  out, we get the Laplace transform of  $T$  in the form:

$$C_2(p, n_1, n_2, \lambda) e^{-\operatorname{tr} \Lambda} t^{\frac{1}{2}n_2 - \frac{1}{2}pn_1} \\ \cdot \int_{B_2 > 0} \int_{\operatorname{Re}(Z) = X_0} e^{\operatorname{tr} Z} |Z|^{-\frac{1}{2}n_1} |I - W|^{-\frac{1}{2}(n_1 + n_2)} |B_2|^{\frac{1}{2}(n_1 + n_2 - p - 1)} \\ \cdot |I + (t\lambda)^{-1} B_2|^{-\frac{1}{2}n_1} \exp(-\operatorname{tr}(I - W))^{-\frac{1}{2}} \Lambda(I - W)^{-\frac{1}{2}} B_2 d B_2 d Z,$$

where  $C_2(p, n_1, n_2, \lambda) = 2^{\frac{1}{2}p(p-1)} \Gamma_p(\frac{1}{2}n_1) / [(2\pi i)^{\frac{1}{2}p(p+1)} \lambda^{\frac{1}{2}pn_1} \Gamma_p(\frac{1}{2}n_2)]$ .

Now letting  $D_2 = \Lambda^{\frac{1}{2}}(I - W)^{-\frac{1}{2}} B_2(I - W)^{-\frac{1}{2}} \Lambda^{\frac{1}{2}}$ , the above expression becomes:

$$(2.2) \quad C_2(p, n_1, n_2, \lambda) e^{-\operatorname{tr} \Lambda} t^{\frac{1}{2}n_2 - \frac{1}{2}pn_1} \int_{D_2 > 0} e^{-\operatorname{tr} D_2} |D_2|^{\frac{1}{2}(n_1 + n_2 - p - 1)} \\ \cdot \int_{\operatorname{Re}(Z) = X_0} e^{\operatorname{tr} Z} |Z|^{-\frac{1}{2}n_1} |I + (t\lambda)^{-1} (I - W) \Lambda^{-1} D_2|^{-\frac{1}{2}n_1} d D_2 d Z.$$

Further, let us write

$$|I + (t\lambda)^{-1} (I - W) \Lambda^{-1} D_2|^{-\frac{1}{2}n_1} = \sum_{k=0}^{\infty} \sum_{\kappa} \left(\frac{1}{2}n_1\right)_{\kappa} (-t)^{-k} \frac{C_{\kappa}(\lambda^{-1} (I - W) \Lambda^{-1} D_2)}{k!}$$

in (2.2) and integrate term by term with respect to  $t$  for sufficiently large values of  $\operatorname{Re}(t)$  and then integrate out  $D_2$  making use of (12) of Constantine [1] to obtain the density of  $T$  in the form:

$$C_2(p, n_1, n_2, \lambda) \Gamma_p\left(\frac{1}{2}(n_1+n_2)\right) e^{-\text{tr } \Omega} |\Lambda|^{-\frac{1}{2}n_1} T^{\frac{1}{2}pn_1-1} \\ \cdot \int_{\text{Re } Z = X_0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\left(\frac{1}{2}n_1\right)_{\kappa} \left(\frac{1}{2}(n_1+n_2)\right)_{\kappa} (-T)^k}{k! \Gamma_p\left(\frac{1}{2}pn_1+k\right)} C_{\kappa}[(\lambda\Lambda)^{-1}(I-W)] dZ$$

which is convergent for  $|T/\lambda\lambda_1| < 1$ , where  $\lambda_1$  is the minimum root of  $\Lambda$  (see [8]). Now  $(\lambda\Lambda)^{-1}$  being symmetric, can be diagonalized by an orthogonal transformation  $H \in O(p)$  and perform the integration over  $O(p)$ , then finally apply (17) of Constantine [2] to get the density function of  $T$  as:

$$(2.3) \quad f(T) = [\Gamma_p\left(\frac{1}{2}(n_1+n_2)\right)/\Gamma_p\left(\frac{1}{2}n_2\right)] |\lambda\Lambda|^{-\frac{1}{2}n_1} e^{-\text{tr } \Omega} T^{\frac{1}{2}pn_1-1} \\ \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\left(\frac{1}{2}(n_1+n_2)\right)_{\kappa} (-T)^k}{k! \Gamma_p\left(\frac{1}{2}pn_1+k\right)} \frac{C_{\kappa}(\lambda^{-1}\Lambda^{-1}) L_{\kappa}^{\frac{1}{2}(n_1-p-1)}(\Omega)}{C_{\kappa}(I)}$$

Formula (2.3) will give special cases as follows:

- a). For  $\Omega = 0$  which implies  $L_{\kappa}^{\gamma}(0) = \left(\frac{1}{2}n_1\right)_{\kappa} C_{\kappa}(I)$  where  $\gamma = \frac{1}{2}(n_1-p-1)$ , we have the result of Khatri [8] formula (9).
- b). For  $\Lambda = I$  and  $\lambda = 1$  we have Theorem 4 of Constantine [2].

3. The moments of  $T$ . In order to obtain the moments of  $T$ , we note that

$$T^k = (\lambda \text{tr } B_1 B_2^{-1})^k = \lambda^k \sum_{\kappa} C_{\kappa}(B_1 B_2^{-1}), \text{ where } B_1 \text{ and } B_2 \text{ are as in Section 2}$$

and we shall use the joint density of  $B_1$  and  $B_2$  in (2.1). The  $k^{\text{th}}$  moment of  $T$  is  $E(T^k)$ . Multiplying (2.1) by  $\lambda^k \sum_{\kappa} C_{\kappa}(B_1 B_2^{-1})$  and integrating out first  $B_2$  using (8) of Khatri [7] and then  $B_1$  using (12) of Constantine [1] we have:

$$E(T^k) = C_1(p, n_2) \lambda^k e^{-\text{tr } \Omega}$$

$$\cdot \int_{\text{Re } Z = X_0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} |I-W|^{-\frac{1}{2}n_1} \sum_{\kappa} \Gamma_p(\frac{1}{2}n_2, \kappa) \Gamma_p(\frac{1}{2}n_1, \kappa) C_{\kappa}[\Lambda(I-W)^{-1}] dZ.$$

Now transform  $\Lambda \rightarrow H \Lambda H'$  where  $H \in O(p)$  and integrate over  $O(p)$ , then the right hand side in the above expression becomes

$$C_1(p, n_2) e^{-\text{tr } \Omega} \sum_{\kappa} \Gamma_p(\frac{1}{2}n_1, \kappa) \Gamma_p(\frac{1}{2}n_2, -\kappa) \frac{C_{\kappa}(\lambda \Lambda)}{C_{\kappa}(I)} \\ \cdot \int_{\text{Re } Z = X_0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} |I-W|^{-\frac{1}{2}n_1} C_{\kappa}((I-W)^{-1}) dZ.$$

Replacing  $W$  by  $\frac{1}{\Omega^2} Z^{-1} \frac{1}{\Omega^2}$  and noting that  $C_{\kappa}((I-W)^{-1}) = C_{\kappa}(I + \Omega(Z-\Omega)^{-1})$  and rearranging the necessary variables, we obtain

$$E(T^k) = C_1(p, n_2) \sum_{\kappa} \Gamma_p(\frac{1}{2}n_1, \kappa) \Gamma_p(\frac{1}{2}n_2, -\kappa) \frac{C_{\kappa}(\lambda \Lambda)}{C_{\kappa}(I)} \\ \cdot \int_{\text{Re } U > 0} e^{\text{tr } U} |U|^{-\frac{1}{2}n_1} C_{\kappa}(I + \Omega U^{-1}) dU,$$

where  $U = Z - \Omega$ . Now using (1.7) of Constantine [2] we have the  $k^{\text{th}}$  moment of  $T$

$$(3.1) \quad E(T^k) = [\Gamma_p(\frac{1}{2}n_2)]^{-1} \sum_{\kappa} \Gamma_p(\frac{1}{2}n_2, -\kappa) C_{\kappa}(\lambda \Lambda) L_{\kappa}^{\frac{1}{2}(n_1-p-1)}(\Omega) / C_{\kappa}(I).$$

Finally using the fact that  $\Gamma_p(t, -\kappa) = [(-1)^k \Gamma_p(t)] / (-t + \frac{1}{2}(p+1))_{\kappa}$ , we get  $k^{\text{th}}$  moment of  $T$  as

$$(3.2) \quad E(T^k) = (-1)^k \sum_{\kappa} \frac{C_{\kappa}(\lambda \Lambda) L_{\kappa}^{\frac{1}{2}(n_1-p-1)}(-\Omega)}{(\frac{1}{2}(p+1-n_2))_{\kappa} C_{\kappa}(I)},$$

which exists only for  $n_2 > 2k+p-1$ .

Formula (3.2) will give special cases for special values of  $\Omega$  and  $\Lambda$ . If we let  $\Omega = 0$ , (3.2) gives

$$E(T^k) = (-1)^k \sum_{\kappa} \left(\frac{1}{2}n_1\right)_{\kappa} C_{\kappa}(\lambda \Lambda) / \left(\frac{1}{2}(p+1-n_2)\right)_{\kappa},$$

which is the result of Khatri [8]; except that his formula contains an error in the denominator. His denominator is  $(\frac{1}{2}(p-n_2-1))_J$  and the right one is  $(\frac{1}{2}(p-n_2+1))_J$ . Substitution of  $\Lambda = I$ ,  $\lambda = 1$  in (3.2) will give the result of Constantine [2] formula (38).

4. Moment generating function of T. Pillai [12] has obtained the joint density function of the roots  $r_1, \dots, r_p$  of  $S_1 S_2^{-1}$  which has the form:

$$(4.1) \quad C(p, n_1, n_2) e^{-\text{tr } \Omega} |\Lambda|^{-\frac{1}{2}n_1} |R|^{\frac{1}{2}(n_1-p-1)} \prod_{i>j} (r_i - r_j) \\ \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_{\kappa}(\Omega)}{(\frac{1}{2}n_1)_{\kappa} C_{\kappa}(I) k!} \sum_{n=0}^{\infty} \sum_{\nu, \delta} \frac{(-1)^n g_{\kappa, \nu}^{\delta} (\frac{1}{2}(n_1+n_2))_{\delta} C_{\delta}(\Lambda^{-1}) C_{\delta}(R)}{C_{\delta}(I) n!},$$

where  $g_{\kappa, \nu}^{\delta}$  are constants (Constantine [2], Pillai and Sugiyama [14]),

$R = \text{diag}(r_1, \dots, r_p)$  with  $0 < r_1 < \dots < r_p < \infty$ ,  $\delta_1 + \dots + \delta_p = k + n$ ,  $\delta = (\delta_1, \dots, \delta_p)$  and

$$(4.2) \quad C(p, n_1, n_2) = \pi^{\frac{1}{2}p^2} \Gamma_p(\frac{1}{2}(n_1+n_2)) / [\Gamma_p(\frac{1}{2}n_1) \Gamma_p(\frac{1}{2}n_2) \Gamma_p(\frac{1}{2}p)]$$



To obtain the m.g.f. of  $T = \lambda \text{tr } S_{12}^{-1}$  we shall take the expected value of  $\exp(t \lambda \text{tr } R)$  with respect to (4.1), since under the transformations employed by Pillai [12] to get (4.1),  $\text{tr } S_{12}^{-1} = \text{tr } R$ . Now transform back  $R \rightarrow H R H'$  where  $H \in O(p)$  and the latter matrix  $R$  is symmetric, then perform the necessary integrations (i.e. use (44) of Constantine [1], and integrate out  $R > 0$ ), we get:

$$E(\exp(t \lambda \text{tr } R)) = C_1(p, n_1, n_2) e^{-\text{tr } \Omega} |\Lambda|^{-\frac{1}{2}n_1} \\ \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_{\kappa}(\Omega)}{(\frac{1}{2}n_1)_{\kappa} k! C_{\kappa}(I)} \sum_{n=0}^{\infty} \sum_{\nu, \delta} \frac{(-1)^n g_{\kappa, \nu}^{\delta} (\frac{1}{2}(n_1+n_2))_{\delta}}{n!} \\ \cdot \int_{R>0} e^{\text{tr}(t\lambda R)} |R|^{\frac{1}{2}(n_1-p-1)} C_{\delta}(\Lambda^{-1}R) dR,$$

where  $C_1(p, n_1, n_2) = \Gamma_p(\frac{1}{2}(n_1+n_2)) / [\Gamma_p(\frac{1}{2}n_1) \Gamma_p(\frac{1}{2}n_2)]$ .

Let us now apply (12) of Constantine [1]. Upon simplification we finally obtain the m.g.f. of  $T$  as:

$$(4.3) \quad E(\exp(t \lambda \text{tr } R)) = [\Gamma_p(\frac{1}{2}(n_1+n_2)) / \Gamma_p(\frac{1}{2}n_2)] e^{-\text{tr } \Omega} | -t \lambda \Lambda |^{-\frac{1}{2}n_1} \\ \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_{\kappa}(\Omega)}{(\frac{1}{2}n_1)_{\kappa} k! C_{\kappa}(I)} \sum_{n=0}^{\infty} \sum_{\kappa, \delta} \frac{(-1)^n (\frac{1}{2}n_1)_{\delta} (\frac{1}{2}(n_1+n_2))_{\delta} g_{\kappa, \nu}^{\delta} C_{\delta}(-(t\lambda)^{-1} \Lambda^{-1})}{n!}$$

Again, as before we can put special values of  $\Omega$  and  $\Lambda$  to get special cases. Making a substitution  $\Omega = 0$  in (4.3) and noting that  $g_{0, \nu}^{\delta} = 1$ ,  $\delta = \nu$  we have:

$$(4.4) \quad E(\exp(t \lambda \text{tr } R)) = [\Gamma_p(\frac{1}{2}(n_1+n_2)) / \Gamma_p(\frac{1}{2}n_2)] | -t \lambda \Lambda |^{-\frac{1}{2}n_1} \\ {}_2F_0(\frac{1}{2}n_1, \frac{1}{2}(n_1+n_2); (t\lambda)^{-1} \Lambda^{-1}),$$

while letting  $\Lambda = I$  and  $\lambda = 1$  and  $v = 0$  which implies  $\kappa = \delta$  we obtain:

$$(4.5) \quad E(\exp(t \operatorname{tr} R)) = [\Gamma_p(\frac{1}{2}(n_1+n_2))/\Gamma_p(\frac{1}{2}n_2)](-t)^{-\frac{1}{2}pn_1} e^{-\operatorname{tr} \Omega} \cdot {}_1F_0(\frac{1}{2}(n_1+n_2); -t^{-1} \Omega)$$

or in an alternative form:

$$(4.6) \quad E(\exp(t \operatorname{tr} R)) = [\Gamma_p(\frac{1}{2}(n_1+n_2))/\Gamma_p(\frac{1}{2}n_2)](-t)^{-\frac{1}{2}pn_1} |I+t^{-1}\Omega|^{-\frac{1}{2}(n_1+n_2)} e^{-\operatorname{tr} \Omega}.$$

5. Moment generating function of  $V^{(p)}$ . To obtain the m.g.f. of Pillai's criterion  $V^{(p)}$  which is defined by  $V^{(p)} = \operatorname{tr}[(\lambda R)(I+\lambda R)^{-1}]$  where  $R = \operatorname{diag}(r_1, \dots, r_p)$ ,  $r_1, \dots, r_p$  being the roots of  $S_1 S_2^{-1}$ , we start from the following joint density of roots of  $S_1 S_2^{-1}$  (Pillai [12]):

$$C_1(p, n_1, n_2) e^{-\operatorname{tr} \Omega} |\Lambda|^{-\frac{1}{2}n_1} |R|^{\frac{1}{2}(n_1-p-1)} |I+\lambda R|^{-\frac{1}{2}(n_1+n_2)} \prod_{i>j} (r_i - r_j) \\ \cdot \int_{\operatorname{Re} Z > 0} e^{\operatorname{tr} Z} |Z|^{-\frac{1}{2}n_1} {}_1F_0(\frac{1}{2}(n_1+n_2); I-\lambda^{-1}(I-W)^{\frac{1}{2}} \Lambda^{-1}(I-W)^{\frac{1}{2}}, \lambda R(I+\lambda R)^{-1}) dZ$$

where

$$C_1(p, n_1, n_2) = \pi^{\frac{1}{2}p^2} \frac{1}{2^{\frac{1}{2}p(p-1)}} \Gamma_p(\frac{1}{2}(n_1+n_2)) / [(2\pi i)^{\frac{1}{2}p(p+1)} \Gamma_p(\frac{1}{2}n_2) \Gamma_p(\frac{1}{2}p)]$$

Let  $L = \lambda R(I+\lambda R)^{-1}$ , then the above density gives the joint density of roots  $\ell_1, \dots, \ell_p$  of  $L$  as:

$$C_1(p, n_1, n_2) e^{-\operatorname{tr} \Omega} |\lambda \Lambda|^{-\frac{1}{2}n_1} |L|^{\frac{1}{2}(n_1-p-1)} |I-L|^{-\frac{1}{2}(n_2-p-1)} \prod_{i>j} (\ell_i - \ell_j) \\ \cdot \int_{\operatorname{Re} Z > 0} e^{\operatorname{tr} Z} |Z|^{-\frac{1}{2}n_1} {}_1F_0(\frac{1}{2}(n_1+n_2); I-\lambda^{-1}(I-W)^{\frac{1}{2}} \Lambda^{-1}(I-W)^{\frac{1}{2}}, L) dZ$$

where  $0 < \ell_1 < \ell_2 < \dots < \ell_p < 1$ .

The expected value of  $\exp(t v^{(p)})$  with respect to this density is

$$C_1(p, n_1, n_2) e^{-\frac{\text{tr } \Omega}{|\lambda \Lambda|} - \frac{1}{2} n_1} \\ \cdot \int_{L > 0}^I e^{t \text{tr } L} |L|^{\frac{1}{2}(n_1 - p - 1)} |I - L|^{\frac{1}{2}(n_2 - p - 1)} \prod_{i > j} (\ell_i - \ell_j) \\ \cdot \int_{\text{Re } Z > 0} e^{\text{tr } Z} |Z|^{-\frac{1}{2} n_1} {}_1F_0\left(\frac{1}{2}(n_1 + n_2); I - \lambda^{-1}(I - W)^{\frac{1}{2}} \Lambda^{-1}(I - W)^{\frac{1}{2}}, L\right) dL dZ.$$

Now use (31) of James [6] and rearrange the order of the integration to get

$$E(\exp(t v^{(p)})) = [2^{\frac{1}{2}p(p-1)} / (2\pi i)^{\frac{1}{2}p(p+1)}] |\lambda \Lambda|^{-\frac{1}{2} n_1} e^{-\frac{\text{tr } \Omega}{|\lambda \Lambda|}} \\ \cdot \int_{S > 0} e^{-\frac{\text{tr } S}{|S|} \frac{1}{2}(n_1 + n_2 - p - 1)} \int_{\text{Re } Z > 0} e^{\text{tr } Z} |Z|^{-\frac{1}{2} n_1} \\ \cdot \int_{L > 0}^I \exp[\text{tr}(t I + S \{I - \lambda^{-1}(I - W)^{\frac{1}{2}} \Lambda^{-1}(I - W)^{\frac{1}{2}}\}) L] \\ \cdot |L|^{\frac{1}{2}(n_1 - p - 1)} |I - L|^{\frac{1}{2}(n_2 - p - 1)} dL dZ dS.$$

After integrating out  $L$  using (47) of James [6] we get:

$$E(\exp(t v^{(p)})) = C_2(p, n_1, n_2) e^{-\frac{\text{tr } \Omega}{|\lambda \Lambda|} - \frac{1}{2} n_1} \int_{\text{Re } Z > 0} e^{\text{tr } Z} |Z|^{-\frac{1}{2} n_1} \\ \cdot \int_{S > 0} e^{-\frac{\text{tr } S}{|S|} \frac{1}{2}(n_1 + n_2 - p - 1)} {}_1F_1\left(\frac{1}{2} n_1; \frac{1}{2}(n_1 + n_2); t I + S \{I - \lambda^{-1}(I - W)^{\frac{1}{2}} \Lambda^{-1}(I - W)^{\frac{1}{2}}\}) dS dZ\right)$$

where

$$C_2(p, n_1, n_2) = [2^{\frac{1}{2}p(p-1)} \Gamma_p(\frac{1}{2}n_1)] / [(2\pi i)^{\frac{1}{2}p(p+1)} \Gamma_p(\frac{1}{2}(n_1+n_2))] .$$

Now expand  ${}_1F_1$  in terms of the series of zonal polynomials and use the relation:

$$(5.1) \quad \frac{C_\kappa(I+A)}{C_\kappa(I)} = \sum_{d=0}^k \sum_{\delta} a_{\kappa, \delta} C_\delta(A) / C_\kappa(I)$$

where  $a_{\kappa, \delta}$  are constants (Constantine [2], Pillai and Jouris [13]), and then integrate out  $S$  to obtain:

$$E(\exp(t v^{(p)})) = [2^{\frac{1}{2}p(p-1)} \Gamma_p(\frac{1}{2}n_1)] / [(2\pi i)^{\frac{1}{2}p(p+1)} |\lambda \Lambda|^{-\frac{1}{2}n_1} e^{-\text{tr } \Omega}]$$

$$\int_{\text{Re } Z > 0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n_1)_\kappa C_\kappa(I)}{k! (\frac{1}{2}(n_1+n_2))_\kappa} \sum_{d=0}^k \sum_{\delta} a_{\kappa, \delta} t^{k-d} (\frac{1}{2}(n_1+n_2))_\delta$$

$$\cdot \frac{C_\delta(I - \lambda^{-1}(I-W)^{\frac{1}{2}} \Lambda^{-1}(I-W)^{\frac{1}{2}})}{C_\delta(I)} dZ .$$

Finally, applying (5.1) and transforming  $\Lambda \rightarrow H \Lambda H'$  and integrating over  $O(p)$  and using (17) of Constantine [2] we obtain the m.g.f. of  $v^{(p)}$  as follows:

$$(5.2) \quad E(\exp(t v^{(p)})) = e^{-\text{tr } \Omega} |\lambda \Lambda|^{-\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n_1)_\kappa C_\kappa(I)}{k! (\frac{1}{2}(n_1+n_2))_\kappa}$$

$$\cdot \sum_{d=0}^k \sum_{\delta} a_{\kappa, \delta} (\frac{1}{2}(n_1+n_2))_\delta t^{k-d} \sum_{n=0}^d \sum_{\nu} \frac{a_{\kappa, \nu} C_\nu(-\lambda^{-1} \Lambda^{-1}) L_\nu^{\frac{1}{2}(n_1-p-1)}(\Omega)}{(\frac{1}{2}n_1)_\nu C_\nu(I) C_\nu(I)} .$$

For  $\Omega = 0$ , (5.2) gives the result of Khatri [8] with a correction of his expression given by Pillai [11]. Another special case which can be derived

from (5.2) is formula (3.5) of Pillai [11] if in (5.2) we let  $\Lambda = I$ ,  $\lambda = 1$  and we use (Constantine [2])

$$(5.3) \quad L_v^{\frac{1}{2}(n_1-p-1)}(\Omega) = \left(\frac{1}{2}n_1\right)_v C_v(I) \sum_{s=0}^n \sum_{\sigma} \frac{(-1)^s a_{v,\delta} C_{\sigma}(\Omega)}{\left(\frac{1}{2}n_1\right)_{\sigma} C_{\sigma}(I)}$$

and finally we let  $\delta = v = \sigma$ .

6. The density function of  $W^{(p)}$ . In terms of the characteristic roots  $r_i$  ( $i = 1, 2, \dots, p$ ) of  $S_1 S_2^{-1}$ , the Wilks' criterion  $W^{(p)}$  is defined by  $W^{(p)} = |I + \lambda R|^{-1}$ . However, in the derivation of the density of  $W^{(p)}$  we consider  $W^{(p)} = |I - L|$ , i.e. we let  $L = \lambda R(I + \lambda R)^{-1}$  in the formula (3.7) Theorem 2 of Pillai [12]. The joint density of characteristic roots  $\ell_1, \ell_2, \dots, \ell_p$  of  $L$ , then, is

$$(6.1) \quad C(p, n_1, n_2) e^{-\text{tr } \Omega} |\lambda \Lambda|^{-\frac{1}{2}n_1} |L|^{\frac{1}{2}(n_1-p-1)} |I-L|^{\frac{1}{2}(n_2-p-1)}$$

$$\cdot \prod_{i>j} (\ell_i - \ell_j) \sum_{k=0}^{\infty} \sum_{\kappa} \left(\frac{n_1+n_2}{2}\right)_{\kappa} \frac{C_{\kappa}(L)}{k!}$$

$$\cdot \sum_{d=0}^k \sum_{\delta} \frac{a_{\kappa,\delta} C_{\delta}(-\lambda^{-1} \Lambda^{-1}) L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\Omega)}{\left(\frac{1}{2}n_1\right)_{\delta} C_{\delta}(I) C_{\delta}(I)}$$

where  $C(p, n_1, n_2)$  is as in (4.2).

To obtain the density function of  $W^{(p)}$ , first we find the  $k^{\text{th}}$  moment of it and then use the results on inverse Mellin transform [3,4,5]. The  $k^{\text{th}}$  moment of  $W^{(p)}$  with respect to (6.1) is

$$\begin{aligned}
E(W^{(p)})^h &= C(p, n_1, n_2) e^{-\text{tr } \Omega} |\lambda \Lambda|^{-\frac{1}{2}n_1} \\
&\cdot \int_{L>0}^I |L|^{\frac{1}{2}(n_1-p-1)} |I-L|^{\frac{1}{2}(n_2+h-p-1)} \prod_{i>j} (\ell_i - \ell_j) \\
&\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \left(\frac{n_1+n_2}{2}\right)_{\kappa} \frac{C_{\kappa}(L)}{k!} (dL) \\
&\cdot \sum_{d=0}^k \sum_{\delta} \frac{a_{\kappa, \delta} C_{\delta}(-\lambda^{-1} \Lambda^{-1}) L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\Omega)}{\left(\frac{1}{2}n_1\right)_{\delta} C_{\delta}(I) C_{\delta}(I)}.
\end{aligned}$$

Now transform  $L \rightarrow H L H'$ , where  $H \in O(p)$  and  $L$  is a symmetric matrix. Integrating out  $H$  using (44) of Constantine [1] and then  $L$  using Theorem 3 of Constantine [1], we obtain the  $h^{\text{th}}$  moment of  $W^{(p)}$  in the form:

$$\begin{aligned}
(6.2) \quad E(W^{(p)})^h &= [\Gamma_p(\frac{1}{2}(n_1+n_2))/\Gamma_p(\frac{1}{2}n_2)] e^{-\text{tr } \Omega} |\lambda \Lambda|^{-\frac{1}{2}n_1} \\
&\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\left(\frac{1}{2}(n_1+n_2)\right)_{\kappa} \left(\frac{1}{2}n_1\right)_{\kappa} C_{\kappa}(I)}{k!} \frac{\prod_{i=1}^p \Gamma(r+b_i)}{\prod_{i=1}^p \Gamma(r+a_i)} \\
&\cdot \sum_{d=0}^k \sum_{\delta} \frac{a_{\kappa, \delta} C_{\delta}(-\lambda^{-1} \Lambda^{-1}) L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\Omega)}{\left(\frac{1}{2}n_1\right)_{\delta} C_{\delta}(I) C_{\delta}(I)}.
\end{aligned}$$

where  $r = \frac{1}{2}n_2 + h - \frac{1}{2}(p-1)$ ,  $b_i = \frac{1}{2}(i-1)$  and  $a_i = \frac{1}{2}n_1 + k_{p-i+1} + b_i$ .

Finally, on (6.2) we apply results on inverse Mellin transform [3,4,5] to obtain the density of  $W^{(p)}$  of the form:



$$\begin{aligned}
(6.3) \quad f(W^{(p)}) &= [\Gamma_p(\frac{1}{2}(n_1+n_2)) / \Gamma_p(\frac{1}{2}n_2)] e^{-\text{tr } \Omega} |\lambda \Lambda|^{-\frac{1}{2}n_1} \\
&\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_{\kappa} (\frac{1}{2}n_1)_{\kappa} C_{\kappa}(I)}{k!} (W^{(p)})^{\frac{1}{2}(n_2-p-1)} \\
&\cdot (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} (W^{(p)})^{-r} \prod_{i=1}^p \Gamma(r+b_i) / \prod_{i=1}^p \Gamma(r+a_i) dr \\
&\cdot \sum_{d=0}^k \sum_{\delta} \frac{a_{\kappa, \delta} C_{\delta}(-\lambda^{-1} \Lambda^{-1}) L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\Omega)}{(\frac{1}{2}n_1)_{\delta} C_{\delta}(I) C_{\delta}(I)}.
\end{aligned}$$

The density function (6.3) can be expressed in terms of Meijer's G-function since the integral in (6.3) is expressible in terms of that function [10]. The density function of  $W^{(p)}$  is therefore:

$$\begin{aligned}
(6.4) \quad f(W^{(p)}) &= [\Gamma_p(\frac{1}{2}(n_1+n_2)) / \Gamma_p(\frac{1}{2}n_2)] e^{-\text{tr } \Omega} |\lambda \Lambda|^{-\frac{1}{2}n_1} (W^{(p)})^{\frac{1}{2}(n_2-p-1)} \\
&\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_{\kappa} (\frac{1}{2}n_1)_{\kappa} C_{\kappa}(I)}{k!} G_{p,p}^{p,0} (W^{(p)} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_p \end{matrix} \right.) \\
&\cdot \sum_{d=0}^k \sum_{\delta} \frac{a_{\kappa, \delta} C_{\delta}(-\lambda^{-1} \Lambda^{-1}) L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\Omega)}{(\frac{1}{2}n_1)_{\delta} C_{\delta}(I) C_{\delta}(I)}.
\end{aligned}$$

As in the previous sections, formula (6.4) will give special cases as follows:

- a) If we let  $\Omega = 0$ ,  $L_{\delta}^{\gamma}(0) = (\frac{1}{2}n_1)_{\delta} C_{\delta}(I)$  where  $\gamma = \frac{1}{2}(n_1-p-1)$ , and after making use of (5.1) we obtain formula (4.7) of Pillai, Al-Ani and Jouris [15] for testing the hypothesis  $H_0: \gamma \Lambda = I$ ,  $\lambda > 0$  being given.

b). For  $\lambda = 1$  and  $\Lambda = I$ , and after making use of (5.3) we let  $\kappa = \delta = \sigma$  ( $\sigma$  being the paratition of  $s$  in the expansion of  $L_\delta$ ), then we have formula (5.2) of [15]. Note that in this case  $\lambda_1, \lambda_2, \dots, \lambda_p$  are the roots of determinantal equation  $|S_1 - \lambda(S_1 + S_2)| = 0$ , and  $w^{(p)}$  is the Wilks' criterion for MANOVA.

7. The density function of the largest root. In this section we derive two expressions for the density function of the largest root  $r_p$  of  $S_1 S_2^{-1}$ . In obtaining this density we start from the joint density of the roots  $r_1, \dots, r_p$  which is given in Pillai [12] by the formula:

$$C_1(p, n_1, n_2) e^{-\text{tr } \Omega} |\Lambda|^{-\frac{1}{2}n_1} |R|^{-\frac{1}{2}(n_1-p-1)} \prod_{i>j}^p (r_i - r_j) \\ \cdot \int_{\text{Re } Z > 0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} \int_{O(p)} 2^{-p} |I + H R H' (I - W)^{-\frac{1}{2}} \Lambda^{-1} (I - W)^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)} dH dZ,$$

where  $R$ ,  $Z$  and  $W$  are as in the previous sections, and

$$C_1(p, n_1, n_2) = 2^{\frac{1}{2}p(p-1)} \Gamma_p(\frac{1}{2}(n_1+n_2)) / [(2\pi i)^{\frac{1}{2}p(p+1)} \Gamma_p(\frac{1}{2}n_2)]$$

Now use lemma 2 of Khatri [8] by taking  $g(F) = r_p$  and also apply formula (44) of Constantine [1], then the second integral in the above expression becomes:

$$[\pi^{\frac{1}{2}p^2} / \Gamma_p(\frac{1}{2}p)] |I + (I - W)^{-\frac{1}{2}} \Lambda^{-1} (I - W)^{-\frac{1}{2}} r_p|^{-\frac{1}{2}(n_1+n_2)} \\ \cdot {}_1F_0(\frac{1}{2}(n_1+n_2); r_p^{-1} (I + (I - W)^{-\frac{1}{2}} \Lambda^{-1} (I - W)^{-\frac{1}{2}} r_p^{-1})^{-1}, r_p I - R)$$

Let  $h_i = r_i/r_p$ ,  $i = 1, 2, \dots, p-1$  and  $H = \text{diag}(h_1, \dots, h_{p-1})$ , then the joint density of  $r_p, h_1, h_2, \dots, h_{p-1}$ , where  $0 < r_p < \infty$ ,  $0 < h_1 < h_2 < \dots < h_{p-1} < 1$  is given by:

$$C_2(p, n_1, n_2) e^{-\text{tr } \Omega} |\Lambda|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} |H|^{\frac{1}{2}(n_1-p-1)} |I_{p-1}-H| \prod_{i>j} (h_i-h_j) \\ \cdot \int_{\text{Re } Z > 0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} |I + (I-W)^{\frac{1}{2}} \Lambda^{-1} (I-W)^{\frac{1}{2}} r_p|^{-\frac{1}{2}(n_1+n_2)} \\ \cdot {}_1F_0\left(\frac{1}{2}(n_1+n_2); (I+(I-W)^{-\frac{1}{2}} \Lambda (I-W)^{-\frac{1}{2}} r_p^{-1})^{-1}, I_{p-1}-H\right) dZ,$$

where  $I_{p-1}$  is the identity matrix of order  $p-1$  and  $I$  is that of order  $p$  and

$$C_2(p, n_1, n_2) = 2^{\frac{1}{2}p(p-1)} \pi^{\frac{1}{2}p^2} \Gamma_p\left(\frac{1}{2}(n_1+n_2)\right) / [(2\pi i)^{\frac{1}{2}p(p+1)} \Gamma_p\left(\frac{1}{2}n_2\right) \Gamma_p\left(\frac{1}{2}p\right)].$$

Expand  ${}_1F_0$  in terms of zonal polynomials and for integration with respect to  $H$  apply lemma 3 of Khatri [8], then we obtain the density function of the largest root  $r_p$  of the form:

$$(7.1) \quad C_3(p, n_1, n_2) e^{-\text{tr } \Omega} |\Lambda|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} \\ \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_{\kappa} \Gamma_{p-1}(\frac{1}{2}p+1, \kappa) C_{\kappa}(I_{p-1})}{k! \Gamma_{p-1}(\frac{1}{2}(n_1+p+1), \kappa) C_{\kappa}(I)} \\ \cdot \int_{\text{Re } Z > 0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} |I + (I-W)^{\frac{1}{2}} \Lambda^{-1} (I-W)^{\frac{1}{2}} r_p|^{-\frac{1}{2}(n_1+n_2)} \\ \cdot C_{\kappa}[(I + (I-W)^{-\frac{1}{2}} \Lambda (I-W)^{-\frac{1}{2}} r_p^{-1})^{-1}] dZ,$$

where

$$C_3(p, n_1, n_2) = C_2(p, n_1, n_2) \Gamma_{p-1}(\frac{1}{2}(n_1-1)) \Gamma_{p-1}(\frac{1}{2}(p-1)) / \pi^{\frac{1}{2}(p-1)^2}.$$

Now, the integral in (7.1) can be written as:

$$\int_{\text{Re } \underline{Z} > 0} e^{\text{tr } \underline{Z}} |\underline{Z}|^{-\frac{1}{2}n_1} |(\underline{I}-\underline{W}) \underline{r}_p^{-1} \underline{\Lambda}|^{-\frac{1}{2}(n_1+n_2)} | \underline{I} + (\underline{I}-\underline{W})^{-\frac{1}{2}} \underline{r}_p^{-1} \underline{\Lambda} (\underline{I}-\underline{W})^{-\frac{1}{2}} |^{-\frac{1}{2}(n_1+n_2)} \\ C_K [ \{ \underline{I} + (\underline{I}-\underline{W})^{-\frac{1}{2}} \underline{r}_p^{-1} \underline{\Lambda} (\underline{I}-\underline{W})^{-\frac{1}{2}} \}^{-1} ] d\underline{Z}.$$

Applying (12) of Constantine [1], the above integral becomes:

$$\frac{|\underline{r}_p^{-1} \underline{\Lambda}|^{-\frac{1}{2}(n_1+n_2)}}{\Gamma_p(\frac{1}{2}(n_1+n_2), \kappa)} \int_{\text{Re } \underline{Z} > 0} e^{\text{tr } \underline{Z}} |\underline{Z}|^{-\frac{1}{2}n_1} |\underline{I}-\underline{W}|^{-\frac{1}{2}(n_1+n_2)} \\ \cdot \int_{\underline{S} > 0} \exp[-\text{tr}\{ \underline{S} (\underline{I} + (\underline{I}-\underline{W})^{-\frac{1}{2}} \underline{r}_p^{-1} \underline{\Lambda} (\underline{I}-\underline{W})^{-\frac{1}{2}} ) \}] |\underline{S}|^{\frac{1}{2}(n_1+n_2-p-1)} \\ \cdot C_K(\underline{S}) d\underline{S} d\underline{Z}.$$

Let  $\underline{B} = (\underline{I}-\underline{W})^{-\frac{1}{2}} \underline{S} (\underline{I}-\underline{W})^{-\frac{1}{2}}$  and compute its Jacobian, which is  $J(\underline{S}; \underline{B}) = |\underline{I}-\underline{W}|^{\frac{1}{2}(p+1)}$ .

After making a substitution and necessary rearrangement, the above expression now can be written as,

$$(7.2) \quad \frac{|\underline{r}_p^{-1} \underline{\Lambda}|^{-\frac{1}{2}(n_1+n_2)}}{\Gamma_p(\frac{1}{2}(n_1+n_2), \kappa)} \sum_{n=0}^{\infty} \sum_v \frac{(-1)^n}{n!} \sum_{\delta} g_{\kappa}^{\delta} \int_{\text{Re } \underline{Z} > 0} e^{\text{tr } \underline{Z}} |\underline{Z}|^{-\frac{1}{2}n_1} \\ \cdot \int_{\underline{B} > 0} \exp[-\text{tr}\{(\underline{I} + \underline{r}_p^{-1} \underline{\Lambda}) \underline{B}\}] e^{\text{tr } \underline{B}} |\underline{B}|^{\frac{1}{2}(n_1+n_2-p-1)} C_{\delta}[(\underline{I}-\underline{W}) \underline{B}] d\underline{B} d\underline{Z},$$

where  $g_{\kappa, \nu}^{\delta}$  are constants (see Khatri and Pillai [9]) and  $\sum \delta_i = k + n$ ,  $\delta = (\delta_1, \dots, \delta_p)$ . The second integral in (7.2) equals

$$(7.3) \quad \int_{\tilde{B} > 0} \sum_{a=0}^{\infty} \sum_{\alpha} \frac{(-1)^a}{a!} \frac{C_{\alpha}(\tilde{I} + r_p^{-1} \Lambda) C_{\alpha}(\tilde{B})}{C_{\alpha}(\tilde{I})} e^{\text{tr } \tilde{B}} |\tilde{B}|^{\frac{1}{2}(n_1 + n_2 - p - 1)} \\ C_{\alpha}[(\tilde{I} - \tilde{W})\tilde{B}] d\tilde{B},$$

where now  $(\tilde{I} + r_p^{-1} \Lambda)$  is a diagonal matrix. Transform  $\tilde{B} \rightarrow \tilde{H} \tilde{B} \tilde{H}'$  where  $\tilde{H} \in O(p)$ , then the volume element

$$(7.4) \quad d\tilde{B} = \prod_{i>j} (b_i - b_j) \prod_{i=1}^p db_i (d\tilde{H}),$$

where  $b_i$ ,  $i = 1, \dots, p$ , are the roots of  $\tilde{B}$ . Making substitution in (7.3) and integrating over  $\tilde{H}$ , (7.3) now becomes:

$$\int \dots \int \sum_{a=0}^{\infty} \sum_{\alpha} \frac{(-1)^a}{a!} \frac{C_{\alpha}(\tilde{I} + r_p^{-1} \Lambda) C_{\alpha}(\tilde{B})}{C_{\alpha}(\tilde{I})} e^{\text{tr } \tilde{B}} |\tilde{B}|^{\frac{1}{2}(n_1 + n_2 - p - 1)} \\ \cdot C_{\delta}(\tilde{B}) \prod_{i>j} (b_i - b_j) \frac{C_{\delta}(\tilde{I} - \tilde{W})}{C_{\delta}(\tilde{I})} \prod_{i=1}^p db_i,$$

where now  $\tilde{B}$  is a diagonal matrix  $\tilde{B} = \text{diag}(b_1, \dots, b_p)$ .

Now let us transform the diagonal matrix  $\tilde{B}$  back to the symmetric matrix  $B$  and use (7.4) then integrate out  $H$ . Then we have

$$\begin{aligned} & \int_{\tilde{B} > 0} \exp[-\text{tr}\{(\tilde{I} + \tilde{r}_p^{-1} \tilde{\Lambda}) \tilde{B}\}] e^{\frac{\text{tr } B}{|B|} \frac{1}{2}(n_1 + n_2 - p - 1)} \\ & \cdot \frac{C_\delta(B) C_\delta(I - W)}{C_\delta(I)} d\tilde{B} \\ & = \frac{C_\delta(I - W)}{C_\delta(I)} \sum_{t=0}^{\infty} \sum_{\tau} \frac{1}{t!} \sum_{\mu} g_{\delta, \tau}^{\mu} \\ & \cdot \int_{\tilde{B} > 0} \exp[-\text{tr}\{(\tilde{I} + \tilde{r}_p^{-1} \tilde{\Lambda}) \tilde{B}\}] |B|^{\frac{1}{2}(n_1 + n_2 - p - 1)} C_{\mu}(B) d\tilde{B}, \end{aligned}$$

where  $g_{\delta, \tau}^{\mu}$  are constants and  $\sum \mu_i = \sum \delta_i + t$ ,  $\mu = (\mu_1, \dots, \mu_p)$ .

Applying (12) of Constantine [1] to the above integral, we see that (7.2) becomes:

$$\begin{aligned} & \frac{|\tilde{I} + \tilde{r}_p \tilde{\Lambda}^{-1}|^{-\frac{1}{2}(n_1 + n_2)}}{\Gamma_p(\frac{1}{2}(n_1 + n_2), \kappa)} \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g_{\kappa, \nu}^{\delta} \int_{\text{Re } Z > 0} e^{\frac{\text{tr } Z}{|Z|} - \frac{1}{2}n_1} C_\delta(I - W) dZ \\ & \cdot [C_\delta(I)]^{-1} \sum_{t=0}^{\infty} \sum_{\tau} \frac{1}{t!} \sum_{\mu} g_{\delta, \tau}^{\mu} \Gamma_p(\frac{1}{2}(n_1 + n_2), \mu) C_{\mu}[(\tilde{I} + \tilde{r}_p^{-1} \tilde{\Lambda})^{-1}]. \end{aligned}$$

Replacing  $W$  by  $\tilde{\Omega}^{\frac{1}{2}} \tilde{Z}^{-1} \tilde{\Omega}^{\frac{1}{2}}$  and making use of (17) of Constantine [2] and combining the result with (7.1), we obtain the density function of the largest root  $r_p$  of  $S_1 S_2^{-1}$  as:



$$\begin{aligned}
(7.5) \quad C(p, n_1, n_2) e^{-\text{tr } \Omega} |\Lambda|^{-\frac{1}{2}n_1} |I + r_p \Lambda^{-1}|^{-\frac{1}{2}(n_1+n_2)} r_p^{\frac{1}{2}pn_1-1} \\
\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}p+1)_{\kappa} (\frac{1}{2}(p-1))_{\kappa}}{k! (\frac{1}{2}(n_1+p+1))_{\kappa} (\frac{1}{2}p)_{\kappa}} \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g_{\kappa, \nu}^{\delta} \frac{L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\Omega)}{(\frac{1}{2}n_1)_{\delta} C_{\delta}(I)} \\
\cdot \sum_{t=0}^{\infty} \sum_{\tau} \frac{1}{t!} \sum_{\mu} g_{\delta, \tau}^{\mu} (\frac{1}{2}(n_1+n_2))_{\mu} C_{\mu}[(I + r_p^{-1} \Lambda)^{-1}] \quad ,
\end{aligned}$$

where

$$(7.6) \quad C(p, n_1, n_2) = \frac{\Gamma(\frac{1}{2}) \Gamma_p(\frac{1}{2}(n_1+n_2)) \Gamma_{p-1}(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p) \Gamma(\frac{1}{2}n_1) \Gamma_p(\frac{1}{2}n_2) \Gamma_{p-1}(\frac{1}{2}(n_1+p+1))} \quad .$$

Note that in obtaining (7.5) and (7.6), the relations  $\Gamma_p(a, \kappa) = (a)_{\kappa} \Gamma_p(a)$  and  $C_{\kappa}(I_{p-1})/C_{\kappa}(I_p) = (\frac{1}{2}(p-1))_{\kappa}/(\frac{1}{2}p)_{\kappa}$  have been used.

The density in (7.5) will give (16) of Khatri [8] if we let  $\Omega = 0$  and  $\kappa = \delta = \mu$ .

Alternately, starting with (7.2) and directly integrating the last integral there using (12) of Constantine [1], we get

$$\begin{aligned}
(7.7) \quad \frac{1}{\Gamma_p(\frac{1}{2}(n_1+n_2), \kappa)} \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g_{\kappa, \nu}^{\delta} \Gamma_p(\frac{1}{2}(n_1+n_2), \delta) \\
\cdot \int_{\text{Re } Z > 0} e^{\text{tr } Z} |Z|^{-\frac{1}{2}n_1} C_{\delta}[(I-W)(r_p \Lambda^{-1})] dZ \quad .
\end{aligned}$$

Diagonalizing  $(r_p \Lambda^{-1})$  by an orthogonal transformation  $H$  and integrating over  $O(p)$  and finally applying (17) of Constantine [2], we obtain the density function of the largest root  $r_p$  of  $S_1 S_2^{-1}$  as:

$$(7.8) \quad C(p, n_1, n_2) e^{-\text{tr } \Omega} |\Lambda|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}p+1)_{\kappa} (\frac{1}{2}(p-1))_{\kappa}}{k! (\frac{1}{2}(n_1+p+1))_{\kappa} (\frac{1}{2}p)_{\kappa}} \\
\cdot \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g_{\kappa, \nu}^{\delta} \frac{(\frac{1}{2}(n_1+n_2))_{\delta} C_{\delta}(r_p \Lambda^{-1}) L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\Omega)}{(\frac{1}{2}n_1)_{\delta} C_{\delta}(I)},$$

where  $C(p, n_1, n_2)$  is as in (7.6).

For  $\Omega = 0$  and  $\kappa = \delta$ , the density of  $r_p$  is

$$(7.9) \quad f(r_p) = C(p, n_1, n_2) |\Lambda|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_{\kappa} (\frac{1}{2}p+1)_{\kappa} (\frac{1}{2}(p-1))_{\kappa} C_{\kappa}(r_p \Lambda^{-1})}{k! (\frac{1}{2}(n_1+p+1))_{\kappa} (\frac{1}{2}p)_{\kappa}} \\
= C(p, n_1, n_2) |\Lambda|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} {}_3F_2\left(\frac{1}{2}(n_1+n_2), \frac{1}{2}p+1, \frac{1}{2}(p-1); \right. \\
\left. \frac{1}{2}(n_1+p+1), \frac{1}{2}p; r_p \Lambda^{-1}\right).$$

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